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Asymptotic Estimates for the Fourier Transform on $\mathbb{R}^n$, $n \geqslant 2$

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INTRODUCTION

We specify conditions on an integrable function f on $\mathbb{R}^n$, $n \geqslant 2$, and the real number α that ensure that its Fourier transform $\hat{f}$ is $O(|x|^{-n-\alpha})$ as $|x| \to \infty$. When $\alpha > 0$, this implies that $\hat{f}$ is integrable. Our interest in this problem was aroused by a result of Madych [5] and its application to the Radon transform [7]. See Madych [5, 6] for references to other results on asymptotics of the Fourier transform.

Motivated by the relationship between the Radon and Fourier transforms, Helgason [3], Smith *et al.* [11], Solmon [12], and Madych and Solmon [7], we look at a function f on $\mathbb{R}^n$ in polar coordinates and specify mixed smoothness and decay conditions on the spherical and radial variables. The main theorems are stated below.

The first result applies to functions of special type.

THEOREM I. *Let* $n \geqslant 2$, $\Omega \in C^n(S^{n-1})$, $g \in C^d((0, \infty))$, *where* $d = (n+2)/2$ *when n is even and* $d = (n+3)/2$ *when n is odd. Let* $0 < \alpha < 1/2$. *Suppose that* $|r^{j-\alpha} g^{(j)}(r)|$ *is bounded for all j such that* $0 \leqslant j \leqslant d$. *If* $f(x) = g(|x|)\, \Omega(x/|x|)$ *is integrable, then* $|\hat{f}(\xi)| \leqslant C(1 + |\xi|)^{-n-\alpha}$.

In the next result, the smoothness conditions on the spherical variable are expressed in terms of the Sobolev spaces $\mathscr{H}^s(S^{n-1})$. We shall often write ess sup (essential supremum) in place of $\| \ \|_\infty$.

THEOREM II. *Let* $n \geqslant 2$. *Let* $0 \leqslant k \leqslant n$ *be an integer and α a real number such that* $0 < k + \alpha < (n-1)/2$. *Assume that* $f \in C^d(\mathbb{R}^n - \{0\})$, $d = n - k$, *is*

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integrable and satisfies $\operatorname{ess\,sup}\{r^{j-\alpha}\,\|(\partial/\partial r)^j f(r\,\cdot)\|_{\mathscr{H}^{s-j}(S^{n-1})}\}<\infty$ *for* $j=0, 1, ..., d$ *and some* $s>n+\alpha$. *Then* $|\hat{f}(\xi)|\leqslant C(1+|\xi|)^{-n-\alpha}$.

When $f(x)=g(|x|)\,\Omega(x/|x|)$ and the functions $r^{j-\alpha}g^{(j)}(r)$ have bounded variation then one can improve on Theorem II.

THEOREM III. *Let* $n\geqslant 2$. *Let* $0\leqslant k\leqslant n$ *be an integer and* α *a real number such that* $0<k+\alpha<(n+1)/2$. *Let* $f(x)=g(|x|)\,\Omega(x/|x|)$ *be integrable where* $\Omega\in\mathscr{H}^s(S^{n-1})$, $s>n+\alpha-1/2$, *and* $g\in C^d((0,\infty))$, $d=n-k$. *If* $r^{j-\alpha}g^{(j)}(r)$ *has bounded variation for* $j=0, 1, ..., d$ *then* $|\hat{f}(\xi)|\leqslant C(1+|\xi|)^{-n-\alpha}$.

We will first prove Theorem I. The proofs of Theorems II and III rest on results on spherical harmonics, Bessel functions, and the Sobolev spaces $\mathscr{H}^s(S^{n-1})$. After reviewing these, we proceed to the proof of the last two theorems.

1. PROOF OF THEOREM I

The proof of Theorem I depends on two lemmas.

LEMMA 1.1. *Let* $\Phi\in C^k(S^{n-1})$ *where* $k=(n-2)/2$ *if* n *is even and* $k=(n-3)/2$ *if* n *is odd. Let*

$$A(r)=\int_{S^{n-1}}\Phi(\omega)\,e^{-ir\langle\omega,\sigma\rangle}\,d\omega,$$

where $\sigma\in S^{n-1}$, $r>0$. *Then*

$$\|r^kA(r)\|_{L^p((0,\infty))}\leqslant C\sum_{j=-1}^{n-3}\|\Phi_j\|_{L^q(S^{n-1})},$$

if $p>(1/2-(n-1)/2q)^{-1}$, *and* $p>q/(q-1)$, *where the* Φ_j *are finite linear combinations of derivatives of* Φ *of order* $\leqslant k$ *and* C *is independent of* σ.

Proof. The proof is based on ideas in [9]. It is sufficient to prove the result when $\sigma=(1,0,...,0)$. Writing $\omega=(s,\sqrt{1-s^2}\gamma)$, $s\in(-1,1)$, $\gamma\in S^{n-2}$, gives

$$\begin{aligned}A(r)&=\int_{-1}^{1}e^{-irs}\left\{(1-s^2)^{(n-3)/2}\int_{S^{n-2}}\Phi(s,\sqrt{1-s^2}\gamma)\,d\gamma\right\}ds\\&=\int_{-\infty}^{\infty}e^{-irs}B(s)\,ds,\end{aligned}$$

where

$$B(s) = (1-s^2)^{(n-3)/2}\,\mathscr{X}(s)\int_{S^{n-2}} \Phi(s, \sqrt{1-s^2}\,\gamma)\,d\gamma,$$

and $\mathscr{X}$ is the characteristic function of $(-1, 1)$. Integration by parts gives

$$r^k A(r) = \int_{-\infty}^{\infty} e^{-irs}\mathscr{X}(s)\sum_{j=-1}^{n-3}\left[h_j(s)(1-s^2)^{j/2}\int_{S^{n-2}} \Phi_j(s, \sqrt{1-s^2}\,\gamma)\,d\gamma\right] ds$$
$$= \int_{-\infty}^{\infty} e^{-irs} G(s)\,ds,$$

where the h_j are polynomials. Let $1/p + 1/p' = 1$, with $p > 2$. Applying the Hausdorf–Young inequality and then Hölder's inequality twice gives

$$\begin{aligned}
&\|r^k A(r)\|^{p'}_{L^p((0,\infty))}\\
&\quad\leqslant \|G\|^{p'}_{L^{p'}(\mathbb{R})}\\
&\quad\leqslant C\sum_{j=-1}^{n-3}\int_{-1}^{1}\left|h_j(s)(1-s^2)^{j/2}\int_{S^{n-2}}\Phi_j(s, \sqrt{1-s^2}\,\gamma)\,d\gamma\right|^{p'} ds\\
&\quad\leqslant C\sum_{j=-1}^{n-3}\int_{-1}^{1} |h_j(s)|^{p'}\,(1-s^2)^{(j-(n-3)/q)\,p'/2}\\
&\qquad\times\left[(1-s^2)^{(n-3)/2}\int_{S^{n-2}}|\Phi_j(s, \sqrt{1-s^2}\,\gamma)|^q\,d\gamma\right]^{p'/q} ds\\
&\quad\leqslant C\sum_{j=-1}^{n-3}\left[\int_{-1}^{1}\left\{|h_j(s)|^{p'}\,(1-s^2)^{(j-(n-3)/q)(p'/2)}\right\}^{(q/(q-p'))} ds\right]^{(q-p')/q}\|\Phi_j\|^{p'}_q\\
&\quad\leqslant C\sum_{j=-1}^{n-3}\|\Phi_j\|^{p'}_q < \infty,
\end{aligned}$$

provided that $p > (1/2 - (n-1)/2q)^{-1}$ and $p > q/(q-1)$.

LEMMA 1.2. *Let g, Φ be two functions defined on* $(0, \infty)$ *and* S^{n-1}, *respectively. Suppose that $\Phi \in C^k(S^{n-1})$ and $|g(r)| \leqslant Cr^{-n+k+\alpha}$, where k is as in Lemma* 1.1 *and* $0 < \alpha < 1/2$. *If $f(x) = g(|x|)\,\Phi(x/|x|)$, $x \in \mathbb{R}^n$, then $|\hat{f}(\xi)| \leqslant C\,|\xi|^{-k-\alpha}$.*

Proof. Let $\xi \neq 0$ and write $\xi = |\xi|\,\xi'$. Writing out the Fourier transform of f in polar coordinates $r\omega$ and then making a change of variable $s = r\,|\xi|$ give

$$|\hat{f}(\xi)| = \left| \int_0^\infty g(r)\, r^{n-1} \int_{S^{n-1}} \Phi(\omega)\, e^{-ir|\xi|\langle \xi', \omega\rangle}\, d\omega\, dr \right|$$
$$= |\xi|^{-n} \left| \int_0^\infty g(s/|\xi|)\, s^{n-k-\alpha} s^k A(s)\, s^{\alpha-1}\, ds \right|$$
$$\leqslant C\, |\xi|^{-k-\alpha} \left[\int_0^1 s^k\, |A(s)|\, s^{\alpha-1}\, ds + \int_1^\infty s^k\, |A(s)|\, s^{\alpha-1}\, ds \right].$$

The first integral above is finite since $\alpha > 0$ and $\|s^k A(s)\|_\infty < \infty$ by Lemma 1.1. Hölder's inequality shows that the second integral is bounded by $\|s^k A(s)\|_p\, [\int_1^\infty s^{(\alpha-1)p'}\, ds]^{1/p'}$, $1/p + 1/p' = 1$. The expression in brackets is finite provided that $(\alpha - 1)\, p' < -1$, while according to Lemma 1.1, $\|s^k A(s)\|_p$ is finite provided that $p > (1/2 - (n-1)/2q)^{-1}$ and $p > q/(q-1)$, where we are free to choose q. Fix $\alpha < 1/2$. By choosing q sufficiently large, we may then take $p > 2$ so that $\alpha p < 1$, and hence so that $(\alpha - 1)\, p' < -1$. Both expressions are then finite and the proof of the lemma is complete.

Proof of Theorem I. Since f is integrable its Fourier transform is bounded. For $v = (v_1, ..., v_n)$ an n-tuple of nonnegative integers, let $|v| = v_1 + \cdots + v_n$ and $D^v = (\partial/\partial x_1)^{v_1} \cdots (\partial/\partial x_n)^{v_n}$. Taking $|v| = d$ and computing in polar coordinates give

$$D^v f = \sum_{|l| \leqslant d} r^{-d+|l|} g^{(|l|)}(r)\, \Omega_l(\omega),$$

where the Ω_l are C^k since Ω is C^n. Recalling $d = n - k$ and applying Lemma 1.2 to each summand give $|(D^v \hat{f})(\xi)| \leqslant C |\xi|^{-k-\alpha}$. On the other hand, $\sum_i |[(\partial/\partial x_i)^d \hat{f}](\xi)| \geqslant n^{-d/2}\, |\xi|^d\, |\hat{f}(\xi)|$. Combining these two estimates completes the proof of Theorem I.

Remark. Only the simple case $q = \infty$ of Lemma 1.1 is needed in the proof of Theorem I. However, the a priori estimate provided by the lemma shows that the result extends to the case where the derivatives (in the sense of distributions) of order $\leqslant n$ of Ω are in $L^q(S^{n-1})$, for q, depending on α, sufficiently large.

2. Spherical Harmonics and Sobolev Spaces on S^{n-1}

For $v = 0, 1, 2, ...$ let $\{Y_{v,j}\}$, $j = 1, 2, ..., m(n, v) < \infty$, denote an orthonormal basis for the harmonic polynomials of degree v on S^{n-1}. The collection $\{Y_{v,j}\}$ forms an orthonormal basis for the Hilbert space $L^2(S^{n-1})$. For s a real number, a distribution u on S^{n-1} is in the Sobolev space $\mathscr{H}^s(S^{n-1})$

if and only if the coefficients $\{a_{\nu,j}\}$ in its expansion in spherical harmonics satisfy

$$\|u\|^2_{\mathscr{H}^s(S^{n-1})} = \sum_{\nu,j} |a_{\nu,j}|^2 \nu^{2s} < \infty,$$

where for simplicity when $\nu = 0$, we take ν^{2s} to be 1 throughout. When $s = l \geqslant 0$ is an integer, then $\mathscr{H}^l(S^{n-1})$ consists of those functions on S^{n-1} whose derivatives (in the sense of distributions) of order $\leqslant l$ are square integrable. Thus $C^l(S^{n-1}) \subset \mathscr{H}^l(S^{n-1})$. Moreover, for all s, the map $D^j_\omega: \mathscr{H}^s(S^{n-1}) \to \mathscr{H}^{s-j}(S^{n-1})$ is continuous for each $j \geqslant 0$, where D^j_ω is a differential operator of order j with C^∞ coefficients. Also, the spaces $\mathscr{H}^s(S^{n-1})$ and $\mathscr{H}^{-s}(S^{n-1})$ are dual. For proofs of the above statements see Mikhlin and Prössdorf [8, pp. 258–263] where the spaces $\mathscr{H}^s(S^{n-1})$ are denoted $D(\delta^{s/2})$.

The following formulas and estimates on spherical and Bessel functions are needed in the sequel.

LEMMA 2.1. (a) $\int_{S^{n-1}} Y_{\nu,j}(\omega)\, e^{-ir\langle\omega,\varphi\rangle}\, d\omega = C_n i^{-\nu} r^{-(n-2)/2} J_{(n-2)/2+\nu}(r)$ $Y_{\nu,j}(\varphi)$, *where* $\varphi \in S^{n-1}$, J_μ *denotes the Bessel function of order* μ, *and* C_n *is a positive constant that depends only on* n.

(b) $\sum_j |Y_{\nu,j}(\omega)|^2 \leqslant C\nu^{n-2}$ *where* C *is a constant that depends only on* n.

Part (a) is proved in [2, p. 215] and (b) in [10, p. 224].

LEMMA 2.2. *If* $R > 1$ *and* $\nu \geqslant 0$, *then*

(a) $|\int_1^R J_\nu(r)\, r^{-\lambda}\, dr| \leqslant C\nu^{-\lambda}$, $\lambda \geqslant -1/2$;

(b) $|\int_1^R J_0(r)\, r^{-\lambda}\, dr| \leqslant C$, $\lambda \geqslant -1/2$, *where* C *is a constant independent of* ν *and* R;

(c) $\int_0^\infty |J_\nu(r)|^2\, r^{-\lambda}\, dr \leqslant C\nu^{-\lambda}$, *if* $\nu > \lambda^2$ *and* $\lambda > 0$, *where* C *is independent of* ν.

See [4] for (a) and (b) and apply Stirling's formula to (2) [13, p. 403] to obtain (c).

3. PROOF OF THEOREMS II AND III

Once appropriate analogous of Lemma 1.2 are established, the proof of Theorems II and III is almost identical to that of Theorem I.

LEMMA 3.1. *Let* $k \geqslant 0$ *be an integer*, α *and* s *real numbers such that*

$0<k+\alpha<(n-1)/2$ and $s>k+\alpha$. Let g be a function on $S^{n-1}\times(0,\infty)$ such that $\operatorname{ess\,sup}\{t^{n-k-\alpha}\|g(t,\cdot)\|_{\mathscr{H}^s(S^{n-1})}\}<\infty$. *Let $h(x)=g(|x|,x/|x|)$, $x\in\mathbb{R}^n$. Then there exists a constant C such that $|\hat{h}(\xi)|\leqslant C|\xi|^{-k-\alpha}$.*

Proof. Fix $\xi\neq 0$, set $r=|x|$, $\omega=x/|x|$, and $\xi=|\xi|\,\xi'$ so that

$$\begin{aligned}\hat{h}(\xi)&=\int_0^\infty\int_{S^{n-1}} g(r,\omega)\,e^{-ir|\xi|\langle\omega,\xi'\rangle}\,d\omega r^{n-1}\,dr\\&=\int_0^\infty\int_{S^{n-1}} r^{n-k-\alpha}g(r,\omega)\,e^{-ir|\xi|\langle\omega,\xi'\rangle}\,d\omega r^{k+\alpha-1}\,dr\\&=\int_0^{1/|\xi|}[\cdots]\,dr+\int_{1/|\xi|}^\infty[\cdots]\,dr=I_1+I_2.\end{aligned}$$

A change of variable $t=r|\xi|$ in I_1 gives

$$\begin{aligned}|I_1|&=|\xi|^{-k-\alpha}\left|\int_0^1\int_{S^{n-1}}(t/|\xi|)^{n-k-\alpha}\,g(t/|\xi|,\omega)\,e^{-it\langle\omega,\xi'\rangle}\,d\omega t^{k+\alpha-1}\,dt\right|\\&\leqslant|\xi|^{-k-\alpha}\operatorname{ess\,sup}\{r^{n-k-\alpha}\|g(r,\cdot)\|_{\mathscr{H}^s(S^{n-1})}\}\\&\quad\times\int_0^1\|e^{-it\langle\xi',\cdot\rangle}\|_{\mathscr{H}^{-s}(S^{n-1})}\,t^{k+\alpha-1}\,dt.\end{aligned}$$

As the map $(t,\xi')\to\exp^{-it\langle\xi',\cdot\rangle}$ is continuous from $\mathbb{R}\times S^{n-1}$ into $\mathscr{H}^s(S^{n-1})$ for all s, the remaining integral is finite and bounded by a constant depending only on s, k, and α provided that $\alpha+k>0$.

To estimate I_2, make the same change of variable as in I_1 and expand g in spherical harmonics; i.e., write $g(r,\omega)=\sum_{\nu,j}a_{\nu,j}(r)\,Y_{\nu,j}(\omega)$. As the functions $Y_{\nu,j}(\omega)\,e^{-it\langle\omega,\xi'\rangle}$ are orthonormal on S^{n-1}, the resulting series converges in L^2 norm for a.e. r and may be integrated term by term over S^{n-1}. Applying Lemma 2.1(a) gives

$$\begin{aligned}|I_2|\leqslant C|\xi|^{-k-\alpha}\Bigg|\int_1^\infty&(t/|\xi|)^{n-k-\alpha}\sum_{\nu,j}a_{\nu,j}(t/|\xi|)\\&\times Y_{\nu,j}(\xi')\,J_{(n-2)/2+\nu}(t)\,t^{k+\alpha-n/2}\,dt\Bigg|.\end{aligned}$$

Repeated application of the Cauchy–Schwarz inequality and Lemma 2.1(b) show that the integral is bounded by

$$\int_1^\infty (t/|\xi|)^{n-k-\alpha} \sum_\nu \left[\sum_j |a_{\nu,j}(t/|\xi|)|^2\right]^{1/2} \nu^{(n-2)/2} |J_{(n-2)/2+\nu}(t)|\, t^{k+\alpha-n/2}\, dt$$
$$\leqslant C \int_1^\infty (t/|\xi|)^{n-k-\alpha} \left[\sum_{\nu,j} |a_{\nu,j}(t/|\xi|)|^2 \nu^{2s}\right]^{1/2}$$
$$\times \left[\sum_\nu |J_{(n-2)/2+\nu}(t)|^2 \nu^{n-2-2s}\right]^{1/2} t^{k+\alpha-n/2+\delta} t^{-\delta}\, dt$$
$$\leqslant C \operatorname*{ess\,sup}_{t>0} \left[t^{n-k-\alpha} \|g(t,\cdot)\|_{\mathscr{H}^s(S^{n-1})}\right] \left(\int_1^\infty t^{-2\delta}\, dt\right)^{1/2}$$
$$\times \left[\sum_\nu \nu^{n-2-2s} \int_1^\infty |J_{(n-2)/2+\nu}(t)|^2\, t^{2(k+\alpha+\delta)-n}\, dt\right]^{1/2}.$$

The integrals are finite provided that $\delta > 1/2$ and $2(k+\alpha+\delta)-n<0$, or letting δ approach 1/2, provided that $\alpha+k<(n-1)/2$. Finally, using the estimate for the remaining integral provided by Lemma 2.2(c) for ν large, the remaining series above converges whenever $s>k+\alpha$. The proof is complete.

Proof of Theorem II. As in the proof of Theorem I it suffices to show that $|(D^\nu f)\hat{}(\xi)| \leqslant C\,|\xi|^{-k-\alpha}$, whenever $|\nu| = d = n-k$. Computation of $D^\nu f$ in polar coordinates $r\omega$ gives

$$D^\nu f(r\omega) = \sum_{|l| \leqslant d} r^{|l|-d} D_\omega^{d-|l|} (\partial/\partial r)^{|l|} f(r\omega).$$

Since $d=n-k$ and $s>n+\alpha$, then $s-d>k+\alpha$. Moreover, for each l

$$r^{n-k-\alpha}\left[r^{|l|-d} \|D_\omega^{d-|l|}(\partial/\partial r)^{|l|} f(r\cdot)\|_{\mathscr{H}^{s-d}(S^{n-1})}\right]$$
$$\leqslant C r^{|l|-\alpha} \|(\partial/\partial r)^{|l|} f(r\cdot)\|_{\mathscr{H}^{s-|l|}(S^{n-1})}.$$

Lemma 3.1 and the hypotheses on f now give that $|(D^\nu f)\hat{}(\xi)| \leqslant C\,|\xi|^{-k-\alpha}$ and the proof of Theorem II is complete.

The lemma needed in the proof of Theorem III is the following.

LEMMA 3.2. *Let $k \geqslant 0$ be an integer and α a real number such that $0<k+\alpha<(n+1)/2$. Assume that $\Omega \in \mathscr{H}^s(S^{n-1})$, $s>k+\alpha-1/2$, and that $h(r)=r^{n-k-\alpha}g(r)$ has bounded variation on $(0,\infty)$. If $f(x)=g(|x|)\,\Omega(x/|x|)$, then $|\hat{f}(\xi)| \leqslant C\,|\xi|^{-k-\alpha}$.*

Proof. Since h is of bounded variation, it is the difference of two nonincreasing functions. Thus without loss of generality, we assume that h itself is bounded and nonincreasing. We first show that

$$\overline{\lim_{R\to\infty}} \left| \int_0^R r^{n-1} g(r) \int_{S^{n-1}} \Omega(\varphi)\, e^{-ir\langle \xi, \varphi\rangle}\, d\varphi\, dr \right|$$
$$\leqslant C\, \|h\|_\infty\, \|\Omega\|_{\mathscr{H}^s(S^{n-1})}\, |\xi|^{-k-\alpha}, \qquad 0<k+\alpha \leqslant (n+1)/2. \quad (3.3)$$

Proceeding as in the proof of Lemma 3.1, it suffices to consider the integral

$$I_2 = \int_{1/|\xi|}^R r^{n-1} g(r) \int_{S^{n-1}} \Omega(\varphi)\, e^{-ir\langle \xi, \varphi\rangle}\, d\varphi\, dr.$$

As before, take $\xi \neq 0$, and write $\xi = |\xi|\, \xi'$. Make a change of variable $t = r\, |\xi|$, expand Ω in spherical harmonics, and integrate the resulting series term by term to obtain

$$C_n\, |\xi|^{-k-\alpha} \sum_{\nu, j} a_{\nu,j} Y_{\nu,j}(\xi') \int_1^{R|\xi|} h(t/|\xi|)\, J_{(n-2)/2+\nu}(t)\, t^{k+\alpha-n/2}\, dt. \quad (3.4)$$

(The interchange of the order of $\int_1^{R|\xi|}$ and $\sum_{\nu,j}$ is justified by the estimate $|J_\mu(t)| \leqslant |t|^\mu / 2^\mu \Gamma(\mu+1)$, $\mu \geqslant -1/2$ [1, 9.1.62] and Fubini's theorem.) Since h is nonincreasing the second mean value theorem and Lemma 2.2(a), (b) show that the integral in (3.4) is bounded by

$$\left| h(1/|\xi|) \int_1^w J_{(n-2)/2+\nu}(t)\, t^{k+\alpha-n/2}\, dt \right| + \left| h(R) \int_w^{R|\xi|} J_{(n-2)/2+\nu}(t)\, t^{k+\alpha-n/2}\, dt \right|$$
$$\leqslant C\, \|h\|_\infty\, \nu^{k+\alpha-n/2}, \qquad k+\alpha \leqslant (n+1)/2.$$

Substituting in (3.4) and applying the Cauchy–Schwarz inequality twice give (3.3).

The hypotheses on h ensure that f is locally integrable. When $k+\alpha < n/2$, f is square integrable on $|x| \geqslant 1$ and for a.e. ξ (and a suitable sequence of values of R)

$$\hat{f}(\xi) = \lim_{R\to\infty} \int_0^R r^{n-1} g(r) \int_{S^{n-1}} \Omega(\varphi) \exp(-ir\langle \xi, \varphi\rangle)\, d\varphi\, dr. \quad (3.5)$$

The proof is complete in this case. It remains to show that the right hand side of (3.5) exists a.e. when $n/2 \leqslant k+\alpha < (n+1)/2$. From Hankel's asymptotic expansion [13, p. 199], for each fixed μ

$$J_\mu(t) = t^{-1/2}(C_1 \cos(t) + C_2 \sin(t)) + O(t^{-3/2}), \qquad t \gg \mu. \quad (3.6)$$

Since h is bounded and nonincreasing and $k+\alpha<(n+1)/2$, substitution of (3.6) into the integral in (3.4) allows one to see that for each ν

$$\lim_{R\to\infty}\int_1^{R|\xi|} h(t/|\xi|)\, J_{(n-2)/2+\nu}(t)\, t^{k+\alpha-n/2}\, dt \qquad \textit{exists.}$$

A posteriori the proof of (3.3) justifies the interchange of limit and summation in (3.4). This establishes the existence of the right hand side of (3.5) for every $\xi\neq 0$ whenever $0<\alpha+k<(n+1)/2$.

Finally, suppose that $0>s>k+\alpha-1/2$ and that $\Omega\in\mathscr{H}^s(S^{n-1})$. Choose a sequence of functions $\{\Omega_j\}\subset L^2(S^{n-1})$ such that $\Omega_j\to\Omega$ in $\mathscr{H}^s(S^{n-1})$. Then the sequence $\{f_j\}=\{g\Omega_j\}$ converges to $f=g\Omega$ in $\mathscr{S}'(\mathbb{R}^n)$, the space of tempered distributions, and hence $\hat{f}_j\to\hat{f}$ in $\mathscr{S}'(\mathbb{R}^n)$. On the other hand, since each $\hat{f}_j$ exists, (3.3) gives that $\{\hat{f}_j\}$ is Cauchy in the weighted L^1 space $L^1(\mathbb{R}^n,(1+|\xi|)^t)$ for any $t<k+\alpha-n$. Hence, choosing an appropriate subsequence if necessary, $\{\hat{f}_j\}$ converges in norm and pointwise a.e. to a function u in this space. By uniqueness of the Fourier transform, $u=\hat{f}$ and clearly $\hat{f}$ satisfies the desired estimate.

With the exception of using Lemma 3.2 in place of Lemma 3.1, the proof of Theorem III is almost identical to that of Theorem II and is omitted.

Remark. In Theorems I, II, III the integrability of f is only used to establish the boundedness of $\hat{f}$ in a neighborhood of the origin. With the assumption of integrability replaced by local integrability, the proofs show that

$$\overline{\lim_{R\to\infty}}\left|\int_{|x|\leqslant R} f(x)\, e^{-i\langle x,\xi\rangle}\, dx\right|\leqslant C\,|\xi|^{-n-\alpha}.$$

Thus, the assumption that $f\in L^1(\mathbb{R}^n)$ can be replaced, for example, by the assumption that $f\in L^1(\mathbb{R}^n)+L^2(\mathbb{R}^n)$ provided the conclusion concerning $\hat{f}$ is replaced by "$\hat{f}$ is locally square integrable and for a.e. ξ, $|\hat{f}(\xi)|\leqslant C\,|\xi|^{-n-\alpha}$."

References

1. I. Abramowitz and M. Stegun, Eds., "Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables," National Bureau of Standards, 1964.
2. A. P. Calderon and A. Zygmund, On a problem of Mihlin, *Trans. Amer. Math. Soc.* **78** (1955), 209–224.
3. S. Helgason, The Radon transform on Euclidean spaces, compact two point homogeneous spaces and Grassmann manifolds, *Acta Math.* **113** (1965), 153–180.
4. L. Lorch and P. Szego, A singular integral whose kernel involves a Bessel function, *Duke Math. J.* **22** (1955), 407–418.

5. W. R. Madych, On Littlewood–Paley functions, *Studia Math.* **50** (1974), 43–63.
6. W. R. Madych, Absolute summability of Fourier transforms on $\mathbb{R}^n$, *Indiana Univ. Math. J.* **25** (1976), 467–479.
7. W. R. Madych and D. C. Solmon, A range theorem for the Radon transform, *Proc. Amer. Math. Soc.*, in press.
8. S. G. Mikhlin and S. Prössdorf, "Singular Integral Operators," Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1986.
9. D. Namazi, A singular integral, *Proc. Amer. Math. Soc.* **96**, No. 3 (1986), 421–424.
10. U. Neri, "Singular Integrals," Springer-Verlag, Berlin/Heidelberg/New York, 1971.
11. K. T. Smith, D. C. Solmon, and S. L. Wagner, Practical and mathematical aspects of the problem of reconstructing objects from radiographs, *Bull. Amer. Math. Soc.* **83** (1977), 1227–1270.
12. D. C. Solmon, Asymptotic formulas for the dual Radon transform and applications, *Math. Z.* **195**, No. 3 (1987), 321–343.
13. G. N. Watson, "A Treatise on the Theory of Bessel Functions," 2nd ed., Macmillan Co., New York, 1944.